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ENERGY EXPRESSIONS FOR ROTATING TAPERED TIMOSHENKO BEAMS

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1. INTRODUCTION

In a recent paper [1], the mass and stiffness matrices of rotating tapered and twisted Timoshenko beams have been derived using standard finite element methods. The authors have subsequently used these matrices to solve the free vibration problem of such beams. The theory developed assumes rectangular cross-section with linearly varying depths and widths and accounts for the effects of shear deformation, rotatory inertia and a given pre-twist. More importantly, it is based on formulating expressions for the strain energy and kinetic energy of the beam element. Of course, it is essential to establish these correctly to ensure high quality stiffness and mass properties of the element.

In this note, the authors show that there are several inaccuracies and sweeping assumptions in both the expressions for strain and kinetic energies presented in reference [1]. The omission of some significant terms in these energy expressions is bound to degrade the performance of the finite element, which may lead to considerable errors in the numerical results. A further aim of this note is to show that the analysis of reference [1], which is restricted to rectangular cross-sections, can be extended to cover a wider range of practical cross-sections.

Figure 1 shows the axis system of a tapered beam of length L with its left-hand end at a distance r_i from the axis of rotation. The beam is assumed to be rotating at a constant angular velocity Ω. In the right-handed Cartesian co-ordinate system chosen, the origin is taken to be at the left-hand end of the beam, as shown, the Y-axis coinciding with the neutral axis of the beam in the undeflected position. The Z-axis is taken to be parallel (but not coincidental) to the axis of rotation, while the X-axis lies in the plane of rotation. The principal axes of the beam cross-section are, therefore, parallel to the X and Z directions. The system is free to flex in the Z direction (flapping) and in the X direction (lead–lag motion). These two motions can be coupled only through Coriolis forces, but for the system shown for the present analysis, this coupling is ignored.

The taper is assumed to be such that

$$A(y) = A_g\left(1 - c\,\frac{y}{L}\right)^n, \tag{1}$$

$$I_{XX}(y) = I_{XX_g}\left(1 - c\,\frac{y}{L}\right)^{n+2}, \tag{2}$$

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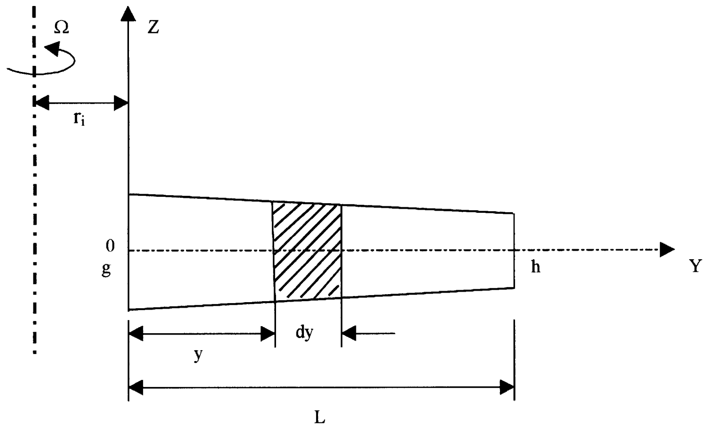

Figure 1. Co-ordinate system and notation for a rotating tapered Timoshenko beam.

$$I_{ZZ}(y) = I_{ZZ_g}\left(1 - c\frac{y}{L}\right)^{n+2}, \tag{3}$$

where A, I_{XX}, I_{ZZ} are the cross-sectional area and second moments of areas about the X and Z axes, respectively, and c is a constant, namely the taper ratio which must be such that $c < 1$ because otherwise the beam tapers to zero between its ends. Values of n of 1 or 2 cover most practical cases, because $n = 1$ gives linear variation of the area of cross-section and a cubic variation of the second moment of area along the length, whereas the corresponding variations for $n = 2$ are the second and fourth orders. Thus, a large number of solid or thin-walled cross-sections can be represented using the values of n as 1 or 2. Samples of cross-sections covered by $n = 1$ and 2 are given in reference [2]. The subscript g denotes a value at g on Figure 1 corresponding to the left-hand end of the tapered beam.Young's modulus E, shear modulus G and density of material ρ, are assumed to be constant so that the mass per unit length ρA, and the bending rigidities EI_{XX} and EI_{ZZ} and the shear rigidity kAG vary according to equations (1)–(3).

From Figure 1, the centrifugal tension $T(y)$ at a distance y from the origin is given by

$$T(y) = \int_y^L \rho A \Omega^2 (r_i + y)\,\mathrm{d}y. \tag{4}$$

Since the area of cross-section for a tapered beam varies along the length, the term ρA should not be treated as constant and taken outside the integral sign. Unfortunately, the authors of reference [1] have made this incorrect assumption in their equation (4). This is clearly incorrect and may lead to unacceptable errors in the results. Assuming the variation of A in the form of equation (1), the above integral can be easily evaluated so that T can be expressed as a function of y.

2. EXPRESSION FOR STRAIN ENERGY

Regarding the correctness of the strain energy expression presented in reference [1], the essential point of this note can be made by simply considering the flexural displacement of the beam in the YZ plane. However, the procedure described below can be easily extended to the flexural displacement in the XY plane as well.

Using the co-ordinate system and notation of Figure 1, the uniform strain $\varepsilon_0(y)$ due to the action of the centrifugal force $T(y)$ along is given by

$$\varepsilon_0(y) = \frac{T(y)}{EA}, \tag{5}$$

where E is Young's modulus and A is the area of cross-section.

The associated axial displacement $u_0(y)$ due to the centrifugal force alone is uniform across the cross-section and follows from

$$u_0'(y) = \varepsilon_0(y) = \frac{T(y)}{EA}, \tag{6}$$

where the prime denotes differentiation with respect to y.

Now introduce a flexural displacement $w(y)$ of the beam neutral axis in the Z direction for an element of length dy between the ordinates y and $(y + dy)$.

Under combined axial and flexural displacements, the element dy will undergo the following deformations. On the left-hand face of the element, a point at a distance $\bar{\eta}$ away from the neutral axis in the Z direction will have the co-ordinates $\{0, (y + u_0 - \bar{\eta}\theta), (\bar{\eta} + w)\}$ whereas the corresponding point on the right-hand face will have the co-ordinates $[0, \{y + u_0 - \bar{\eta}\theta + (1 + u_0' - \bar{\eta}\theta')\,\mathrm{d}y\}, (\bar{\eta} + w + w'\,\mathrm{d}y)]$.

Thus the direct strain of the element at a distance $\bar{\eta}$ from the neutral axis, due to bending and stretching is given by

$$\varepsilon(y, \eta) = [(1 + u_0' - \bar{\eta}\theta')^2 + (w')^2]^{1/2} - 1 \cong u_0' - \bar{\eta}\theta' + \frac{1}{2}(w')^2. \tag{7}$$

Assuming the section rotation θ, the shearing strain γ induced in the element is given by

$$\gamma = w' - \theta. \tag{8}$$

The strain energy due to flexure U_f then follows as

$$U_f = \iiint_V \frac{E\varepsilon^2}{2}\,\mathrm{d}V = \frac{E}{2}\int_A \int_0^L \left\{u_0' - \bar{\eta}\theta' + \frac{1}{2}(w')^2\right\}^2 \mathrm{d}y\,\mathrm{d}A. \tag{9}$$

Since the neutral axis passes through the centroid and the area (A) and the second moment of area (I_{XX}) of the cross-section are, respectively, given by $A(y) = \int_A \mathrm{d}A$ and $I_{XX}(y) = \int_A \bar{\eta}^2\,\mathrm{d}A$, U_f can be simplified as (for simplicity, I_{XX} is hereafter denoted by I)

$$U_f = \frac{1}{2}\int_0^L EA(u_0')^2\,\mathrm{d}y + \frac{1}{2}\int_0^L EI(\theta')^2\,\mathrm{d}y + \frac{1}{2}\int_0^L EAu_0'(w')^2\,\mathrm{d}y. \tag{10}$$

Using equations (4)–(6) and expressing u_0' in terms of the centrifugal tension $T(y)$

$$U_f = C_1 + \frac{1}{2}\left[\int_0^L EI(\theta')^2\,\mathrm{d}y + \int_0^L T(w')^2\,\mathrm{d}y\right], \tag{11}$$

where C_1 is a constant and T is given by equation (4).

The strain energy due to shear U_s is given by

$$U_s = \iiint_V \frac{G\gamma^2}{2}\,\mathrm{d}V = \frac{1}{2}\int_A \int_0^L G\gamma^2\,\mathrm{d}A\,\mathrm{d}y = \frac{1}{2}kAG\int_0^L \gamma^2\,\mathrm{d}y, \tag{12}$$

where G is the shear modulus, k the section shape factor so that kAG is the shear rigidity of the beam cross-section.

Substituting the expression for γ from equation (8) into equation (12) gives

$$U_2 = \frac{1}{2}\int_0^L kAG(w' - \theta)^2\,\mathrm{d}y. \tag{13}$$

The total strain energy $\mathscr{U}$ of the beam is thus

$$\mathscr{U} = U_f + U_s = \frac{1}{2}\left[\int_0^L \{EI(\theta')^2 + T(w')^2 + kAG(w' - \theta)^2\}\,\mathrm{d}y\right] + C_1. \tag{14}$$

3. EXPRESSION FOR KINETIC ENERGY

The kinetic energy of the rotating Timoshenko beam element is derived from the velocity components of a point at a distance $\bar{\eta}$ from the neutral axis. From Figure 1, the three components of the velocities of this point in the X, Y and Z directions are, respectively, given by

$$V_x = -\Omega(y + u_0 - \bar{\eta}\theta), \quad V_y = -\bar{\eta}\dot{\theta}, \quad V_z = \dot{w} \tag{15}$$

so that the kinetic energy $\mathscr{T}$ of the rotating Timoshenko beam is

$$\mathscr{T} = \frac{1}{2}\int_A \int_0^L (V_x^2 + V_y^2 + V_z^2)\,\rho\,\mathrm{d}A\,\mathrm{d}y. \tag{16}$$

Substituting equation (15) into equation (16) gives

$$\mathscr{T} = \frac{1}{2}\left[\int_A \int_0^L \rho\Omega^2 (y + u_0)^2\,\mathrm{d}y\,\mathrm{d}A + \int_A \int_0^L \rho\Omega^2\,\bar{\eta}^2\theta^2\,\mathrm{d}y\,\mathrm{d}A - \int_A \int_0^L 2\rho\Omega^2\,\bar{\eta}\,(y + u_0)\,\theta\,\mathrm{d}y\,\mathrm{d}A \right.$$
$$\left. + \int_A \int_0^L \rho\bar{\eta}^2\dot{\theta}^2\,\mathrm{d}y\,\mathrm{d}A + \int_A \int_0^L \rho\dot{w}^2\,\mathrm{d}y\,\mathrm{d}A\right]. \tag{17}$$

The first integral is constant and the third one is zero so that $\mathscr{T}$ takes the following simplified form:

$$\mathscr{J} = C_2 + \frac{1}{2}\left[\int_0^L \rho I(\Omega^2\theta^2 + \dot{\theta}^2)\,\mathrm{d}y + \int_0^L \rho A\dot{w}^2\,\mathrm{d}y\right], \tag{18}$$

where C_2 is a constant

It is significant to note that the term $\int_0^L \rho I\Omega^2\theta^2\,\mathrm{d}y$ is important, particularly for higher rotational speed Ω, but has been omitted by the authors of reference [1] as well as by some recent investigators [3, 4]. The importance of this term for the case of rotating uniform Timoshenko beams has been demonstrated in a recent paper [5]. It is considered that the term cannot be ignored for the case of rotating tapered Timoshenko beams. Further development of the dynamic motion of the beam then follows from processing the Lagrangian $\mathscr{L} = \mathscr{T} - \mathscr{U}$. Only with a correct formulation of $\mathscr{L}$, it is possible to proceed safely to a numerical model.

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